Math 255A Lecture 4 Notes

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1 The Spanning Criterion and Runge's Theorem

1.1 The spanning criterion

For the complex geometric Hahn-Banach theorem, we don't actually need the assumption that C is balanced.

Theorem 1.1 (complex geometric Hahn-Banach)**.** *Let V be a complex normed vector space, and let $C \subseteq V$ be open, convex, and nonempty. Let $x_0 \notin C$. Then there is a continuous linear map $f : V \to \mathbb{C}$ such that $f(x) \neq f(x_0)$.*

Proof. We can regard V as a vector space over $\mathbb{R}$. Then there exists a continuous $\mathbb{R}$-linear $f_1 : V \to \mathbb{R}$ such that $f_1(x) < f_1(x_0)$ for all $x \in C$. We set $f(x) = f_1(x) - if_1(ix)$. This is $\mathbb{C}$-linear, continuous, and $f(x) \neq f(x_0)$. $\square$

Corollary 1.1. *Let $A \subseteq V$ be closed, convex, and nonempty. Let $x \notin A$. Then there exists a linear continuous $f : V \to \mathbb{C}$ such that $\inf_{y \in A} |f(y) - f(x)| > 0$.*

We will return to the idea of a balanced set later, so our previous discussion is not a waste.

Theorem 1.2 (spanning criterion)**.** *Let V be a normed vector space over $\mathbb{C}$, and let W be a linear subspace. Then the closure $\overline{W}$ can be described as follows:*

$$\overline{W} = \{v \in V : f(v) = 0 \text{ for all } f \in V^* \text{ s.t. } f|_W = 0\}.$$

In other words,

$$\overline{W} = \bigcap_{\substack{f \in V^* \\ f|_W = 0}} \ker(f).$$

Proof. ($\subseteq$): If f is linear and continuous with $f|_W = 0$, then $f|_{\overline{W}} = 0$. So $\overline{W} \subseteq \ker(f)$.

($\supseteq$): Let $x \notin \overline{W}$. $\overline{W}$ is closed and convex, so there exists a continuous linear form $f : V \to \mathbb{C}$ such that $f(x) \neq f(y)$ for all $y \in \overline{W}$. In particular, $f(x) \neq 0$. Let $y \in \overline{W}$. Then $\lambda y \in \overline{W}$ for all $\lambda \in \mathbb{C}$. So $f(x) \neq \lambda f(y)$ for all λ. Thus, $f(y) = 0$ for all $y \in \overline{W}$. We get $f|_W = 0$ and $f(x) \neq 0$. $\square$

Remark 1.1. We can get the exact same statement in the real case, as well.

1.2 Runge's theorem

We will have two types of applications of the Hahn-Banach theorem:

1. approximation theorems
2. existence theorems.

Theorem 1.3 (Runge). *Let $K \subseteq \mathbb{C}$ be a compact set with $K^c = \mathbb{C} \setminus K$ connected. Let f be a function which is holomorphic in a neighborhood of K. Then for any $\varepsilon > 0$, there exists a holomorphic polynomial g such that $|f(z) - g(z)| \le \varepsilon$ for all $z \in K$.*

Before we prove this, let's mention a fact from complex analysis that we will need in the proof.

Proposition 1.1. *Let $\omega \subseteq \mathbb{C}$ be a bounded, open set with C^1-boundary and let $u \in C^1(\overline{\omega})$. Then*

$$u(z) = \frac{1}{2\pi i} \int_{\partial\omega} \frac{u(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi} \iint_{\omega} \frac{\partial u}{\partial \overline{\zeta}}(\zeta) \frac{1}{\zeta - z} L(d\zeta),$$

where $L(d\zeta)$ is Lebesgue measure in $\mathbb{C}$, and

$$\frac{\partial}{\partial \overline{\zeta}} = \frac{1}{2} \left(\frac{\partial}{\partial \operatorname{Re}(\zeta)} + i \frac{\partial}{\partial \operatorname{Im}(\zeta)} \right)$$

is the Cauchy-Riemann operator.

Proof. Here is the idea. Apply the Stokes-Green formula to the function $\zeta \mapsto u(\zeta)/(\zeta - z)$ in $\omega_\varepsilon = \{\zeta \in \omega : |\zeta - z| > \varepsilon\}$:

$$\int_{\partial\omega_\varepsilon} \frac{u(\zeta)}{\zeta - z}\, d\zeta = 2i \iint_{\omega_\varepsilon} \underbrace{\frac{\partial}{\partial \overline{\zeta}} \left(\frac{u(\zeta)}{\zeta - z} \right)}_{\frac{1}{\zeta - z} \frac{\partial u}{\partial \overline{\zeta}}} L(d\zeta)$$

and let $\varepsilon \to 0^+$. $\square$

Proof. Apply the spanning criterion with $V = C(K)$ (equipped with the sup norm) and $W = \{p|_K : p \text{ is a polynomial}\}$. Let f be holomorphic in a neighborhood of K. To show that $f|_K \in \overline{W}$, we need to show that if $L \in C(K)^*$ satisfies $L(p) = 0$ for all polynomials p, then $L(f) = 0$. By the Riesz representation theorem, the dual of $C(K)$ is the space of (Radon) measures on K. We have to show that if μ is a measure on K such that $\int_K z^n\, d\mu(z) = 0$ for all n, then $\int_K f(z)\, d\mu(z) = 0$.

Now let $f \in \mathrm{Hol}(\omega)$, where ω is a neighborhood of K. Let $\psi \in C_0^1(\omega)$ (the set of C^1 functions on ω with compact support) be such that $\psi = 1$ near K. Apply the proposition to $u = f\psi \in C_0^1(\mathbb{C})$. Then

$$f(z)\psi(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} f(\zeta) \frac{\partial \psi}{\partial \bar{\zeta}}(\zeta) \frac{1}{\zeta - z} \, L(d\zeta)$$

for all $z \in K$.

Consider

$$\begin{aligned} \int_K f(z) \, d\mu(z) &= \int_K \left(-\frac{1}{\pi} \iint_{\mathbb{C}} f(\zeta) \frac{\partial \psi}{\partial \bar{\zeta}}(\zeta) \frac{1}{\zeta - z} \, L(d\zeta) \right) d\mu(z) \\ &= -\frac{1}{\pi} \iint_{\mathbb{C} \setminus K} \frac{\partial \psi}{\partial \bar{\zeta}}(\zeta) f(\zeta) \left(\int_K \frac{1}{\zeta - z} \, d\mu(z) \right) L(d\zeta). \end{aligned}$$

If suffices to show that

$$\int_K \frac{1}{\zeta - z} \, d\mu(z) = 0,$$

where $\zeta \in \mathbb{C} \setminus K$. We will finish this next time. $\square$